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Positive and negative solutions of a boundary value problem for a fractional q,ω -difference equation

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ABSTRACT : *In this work, we study a boundary value problem for a fractional* q,ω *-difference equation. By using the monotone iterative technique and lower-upper solution method, we get the existence of positive or negative solutions under the nonlinear term is local continuity and local monotonicity. The results show that we can construct two iterative sequences for approximating the solutions.*

KEYWORDS: *fractional* q,ω *-difference equation; positive and negative solutions; lower and upper solutions; iterative method;*

I. INTRODUCTION

A quantum calculus substitutes the classical derivative by a difference operator, which allows one to deal with sets of non-differentiable functions. There are many different types of quantum difference operators such as h-calculus, q-calculus, Hahn's calculus. These operators are also found in many applications of mathematical areas such as orthogonal polynomials, combinatorics and the calculus of variations.

Hahn [1] introduced his difference operator $D_{q,\omega}$ as follows:

$$D_{q,\omega} f(t) = \frac{f(qt+\omega) - f(t)}{t(q-1)+\omega}, \qquad t \neq \frac{\omega}{1-q},$$

where f is real function, and $q \in (0,1)$ and $\omega > 0$ are real fixed numbers. Malinowska and Torres [2,3] introduced the Hahn quamtum variational calculus, while Malinowska and Martins [4] investigated the generalized transversality conditions for the Hahn quantum variational calculus. Recently, Hamza et al. [5,6] studied the theory of linear Hahn difference equations, and studied the existence and uniqueness results of the initial value problems with Hahn difference equations using the method of successive approximations.

Motivated by the aforementioned work, we consider the following nonlinear boundary value problem for a fractional q,ω -difference equation:

$$\begin{cases} {}_aD_{q,\omega}^{\alpha} u(t) + f(t,u(t)) = 0, t \in (a,b), \\ u(a) = {}_aD_{q,\omega} u(a) = {}_aD_{q,\omega} u(b) = 0, \end{cases} \tag{1.1}$$

where $q \in (0,1)$, $2 < \alpha \leq 3$, $f:[a,b]\times[0,+\infty)$, ${}_aD_{q,\omega}^{\alpha}$ is the fractional q,ω -derivative of the Riemann - Liouville type?

II. BACKGROUND AND DEFINITIONS

To show the main result of this work, we give in the following some basic definitions, lemmas and theorems, which can be found in [7].

Definition 2.1 [7] *Let I be a closed interval of* $\mathbb{R}$ *such that* $\omega_0, a, b \in I$. *For* $f : I \to \mathbb{R}$ *we define the* q,ω*-integral of* f *from a to b by*

$$\int_a^b f(t)d_{q,\omega}t := \int_{\omega_0}^b f(t)d_{q,\omega}t - \int_{\omega_0}^a f(t)d_{q,\omega}t,$$

where

$$\int_{\omega_0}^x f(t)d_{q,\omega}t := (x(1-q)-\omega)\sum_{k=0}^{\infty} q^k f(xq^k + [k]_{q,\omega}), \qquad x \in I,$$

with $[k]_{q,\omega} = \dfrac{\omega(1-q^k)}{1-q}$ *for* $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, *provided that the series converges at* $x = a$ *and* $x = b.$

Lemma 2.2 [7] *Assume* $f : I \to \mathbb{R}$ *be continuous at* ω_0. *Define* $F(x) := \int_{\omega_0}^x f(t)d_{q,\omega}t$. *Then* F *is continuous at* ω_0. *Futhermore,* $D_{q,\omega}F(x)$ *exists for every* $x \in I$ *and* $D_{q,\omega}F(x) = f(x)$. *Conversely,*

$$\int_a^b D_{q,\omega}F(x)d_{q,\omega}t = f(b) - f(a),$$

for all $a, b \in I$.

Lemma 2.3 [7] *For* $\alpha \in \mathbb{R}$,

$$\Gamma_{q,\omega}(\alpha+1) = [\alpha]_q \Gamma_{q,\omega}(\alpha), \qquad \Gamma_{q,\omega}(1) = 1.$$

For any positive integer k,

$$\Gamma_{q,\omega}(k+1) = [k]!.$$

Definition 2.4 [7] *Let* $\alpha \geq 0$ *and* f *be a function defined on* [a,b]. *The Hahn's fractional integration of Riemann-Liouville type is given by* $({}_aI_{q,\omega}^0 f)(t) = f(t)$ *and*

$$({}_aI_{q,\omega}^\alpha f)(t) = \frac{1}{\Gamma_q(\alpha)}\int_a^t (t - {}_{\omega_0}\Phi_q(s))_{\omega_0}^{(\alpha-1)} f(s)d_{q,\omega}s, \qquad \alpha > 0, t \in [a,b].$$

Definition 2.5 [7] *The fractional* q,ω*-derivative of the Riemann -Liouville type is*

$$({}_aD_{q,\omega}^\alpha f)(t) = (D_{q,\omega}^{\lceil\alpha\rceil}\, {}_aI_{q,\omega}^{\lceil\alpha\rceil-\alpha} f)(t), \qquad \alpha > 0,$$

where $\lceil\alpha\rceil$ *denotes the smallest integer greater or equal to* α.

Theorem 2.6 [7] *Let* $\alpha \in (N-1, N]$. *Then for some constants* $c_i \in \mathbb{R}$, $i = 1, 2, \cdots, N$, *the following equality holds:*

$$({}_aI_{q,\omega}^\alpha\, {}_aD_{q,\omega}^\alpha f)(x) = f(x) + c_1(x-a)_{\omega_0}^{(\alpha-1)} + c_2(x-a)_{\omega_0}^{(\alpha-2)} + \cdots + c_N(x-a)_{\omega_0}^{(\alpha-N)}.$$

Lemma 2.7 [8] *Assume that X is a Banach space and K is a normal cone in X,* $T:[u_0,v_0]\to X$ *is a completely continuous increasing operator which satisfies* $u_0\le Tu_0, Tv_0\le v_0$. *Then T has a minimal fixed point* u_* *and a maximal fixed point* v^* *with* $u_0\le u_*\le v^*\le v_0$. *In addition,*

$$u_*=\lim_{n\to\infty}T^n u_0,\qquad v^*=\lim_{n\to\infty}T^n v_0,$$

where $\{T^n u_0\}_{n=1}^{\infty}$ *is an increasing sequence,* $\{T^n v_0\}_{n=1}^{\infty}$ *is a decreasing sequence.*

III. EXISTENCE OF q,ω-FRACTIONAL POSITIVE SOLUTIONS FOR PROBLEM

Lemma 3.1 *Assume* $g\in C[a,b]$, *then the following boundary value problem:*

$$\begin{cases} {}_aD^{\alpha}_{q,\omega}u(t)+g(t)=0, t\in(a,b),\\ u(a)={}_aD_{q,\omega}u(a)={}_aD_{q,\omega}u(b)=0,\end{cases}$$

has a unique solution

$$u(t)=\int_a^b G(t,{}_{\omega_0}\Phi_q(s))g(s)d_{q,\omega}s,$$

where

$$G(t,{}_{\omega_0}\Phi_q(s))=\frac{1}{\Gamma_q(\alpha)}\begin{cases}\dfrac{(t-a)^{(\alpha-1)}_{\omega_0}}{(b-a)^{(\alpha-2)}_{\omega_0}}(b-{}_{\omega_0}\Phi_q(s))^{(\alpha-2)}_{\omega_0}-(t-{}_{\omega_0}\Phi_q(s))^{(\alpha-1)}_{\omega_0},\\ a\le {}_{\omega_0}\Phi_q(s)\le t\le b,\\ \dfrac{(t-a)^{(\alpha-1)}_{\omega_0}}{(b-a)^{(\alpha-2)}_{\omega_0}}(b-{}_{\omega_0}\Phi_q(s))^{(\alpha-2)}_{\omega_0}, \qquad a\le t\le {}_{\omega_0}\Phi_q(s)\le b.\end{cases}$$

Proof In view of Theorem 2.6,

$$u(t)=-\frac{1}{\Gamma_q(\alpha)}\int_a^t(t-{}_{\omega_0}\Phi_q(s))^{(\alpha-1)}_{\omega_0}g(s)d_{q,\omega}s+c_1(x-a)^{(\alpha-1)}_{\omega_0}+c_2(x-a)^{(\alpha-2)}_{\omega_0}$$
$$+c_3(x-a)^{(\alpha-3)}_{\omega_0},$$

Since $u(a)={}_aD_{q,\omega}u(a)=0,$ we have $c_2=c_3=0.$ From the boundary condition ${}_aD_{q,\omega}u(b)=0,$ we get

$$c_1=\frac{1}{\Gamma_q(\alpha)(b-a)^{(\alpha-2)}_{\omega_0}}\int_a^b(b-{}_{\omega_0}\Phi_q(s))^{(\alpha-2)}_{\omega_0}g(s)d_{q,\omega}s.$$

Hence

$$u(t)=-\frac{1}{\Gamma_q(\alpha)}\int_a^t(t-{}_{\omega_0}\Phi_q(s))^{(\alpha-1)}_{\omega_0}g(s)d_{q,\omega}s+\frac{(t-a)^{(\alpha-1)}_{\omega_0}}{(b-a)^{(\alpha-2)}_{\omega_0}\Gamma_q(\alpha)}$$
$$\times\int_a^b(b-{}_{\omega_0}\Phi_q(s))^{(\alpha-2)}_{\omega_0}g(s)d_{q,\omega}s$$

$$=\frac{1}{\Gamma_q(\alpha)}\int_a^t\left[\frac{(t-a)_{\omega_0}^{(\alpha-1)}}{(b-a)_{\omega_0}^{(\alpha-2)}}(b-{}_{\omega_0}\Phi_q(s))_{\omega_0}^{(\alpha-2)}-(t-{}_{\omega_0}\Phi_q(s))_{\omega_0}^{(\alpha-1)}\right]g(s)d_{q,\omega}s$$

$$+\frac{1}{\Gamma_q(\alpha)}\int_t^b\left[\frac{(t-a)_{\omega_0}^{(\alpha-1)}}{(b-a)_{\omega_0}^{(\alpha-2)}}(b-{}_{\omega_0}\Phi_q(s))_{\omega_0}^{(\alpha-2)}\right]g(s)d_{q,\omega}s$$

$$=\int_a^b G(t,{}_{\omega_0}\Phi_q(s))g(s)d_{q,\omega}s.$$

□

Lemma 3.2 *The function* $G(t,{}_{\omega_0}\Phi_q(s))$ *has the following properties:*

(a) $G(t,{}_{\omega_0}\Phi_q(s))\ge 0, G(t,{}_{\omega_0}\Phi_q(s))\le G(b,{}_{\omega_0}\Phi_q(s)),\quad a\le t,{}_{\omega_0}\Phi_q(s)\le b;$

(b) $G(t,{}_{\omega_0}\Phi_q(s))\ge \dfrac{(t-a)_{\omega_0}^{(\alpha-1)}}{(b-a)_{\omega_0}^{(\alpha-2)}}G(b,{}_{\omega_0}\Phi_q(s)),\quad a\le t,{}_{\omega_0}\Phi_q(s)\le b.$

Remark 3.3 *The function* $G(t,{}_{\omega_0}\Phi_q(s))$ *has some other properties:*

(1) $G(t,{}_{\omega_0}\Phi_q(s))\le \dfrac{1}{\Gamma_q(\alpha)}\dfrac{(t-a)_{\omega_0}^{(\alpha-1)}}{(b-a)_{\omega_0}^{(\alpha-2)}}(b-{}_{\omega_0}\Phi_q(s))_{\omega_0}^{(\alpha-2)}\le\dfrac{1}{\Gamma_q(\alpha)}(t-a)_{\omega_0}^{(\alpha-1)},$

$a\le t,{}_{\omega_0}\Phi_q(s)\le b.$

(2) *According to the property of being non-decreasing of function*

$$\frac{(b-{}_{\omega_0}\Phi_q(s))_{\omega_0}^{(\alpha-2)}}{(b-a)_{\omega_0}^{(\alpha-2)}}=\frac{(b-{}_{\omega_0}\Phi_q^{i+1}(s))(b-{}_{\omega_0}\Phi_q^{i+\alpha-2}(a))}{(b-{}_{\omega_0}\Phi_q^{i+\alpha-1}(s))(b-{}_{\omega_0}\Phi_q^{i}(a))}$$

on a and non-increasing of $(t_2-{}_{\omega_0}\Phi_q(s))_{\omega_0}^{(\alpha-2)}$ *on s, We can obtain the following inequalities:*

(a) *For* $a\le t_1\le t_2\le{}_{\omega_0}\Phi_q(s)\le b,$ *we get*

$$\left|G(t_2,{}_{\omega_0}\Phi_q(s))-G(t_1,{}_{\omega_0}\Phi_q(s))\right|$$

$$=\left|\frac{1}{\Gamma_q(\alpha)}\frac{(t_2-a)_{\omega_0}^{(\alpha-1)}}{(b-a)_{\omega_0}^{(\alpha-2)}}(b-{}_{\omega_0}\Phi_q(s))_{\omega_0}^{(\alpha-2)}-\frac{1}{\Gamma_q(\alpha)}\frac{(t_1-a)_{\omega_0}^{(\alpha-1)}}{(b-a)_{\omega_0}^{(\alpha-2)}}(b-{}_{\omega_0}\Phi_q(s))_{\omega_0}^{(\alpha-2)}\right|$$

$$=\frac{1}{\Gamma_q(\alpha)}\frac{(b-{}_{\omega_0}\Phi_q(s))_{\omega_0}^{(\alpha-2)}}{(b-a)_{\omega_0}^{(\alpha-2)}}\left|(t_2-a)_{\omega_0}^{(\alpha-1)}-(t_1-a)_{\omega_0}^{(\alpha-1)}\right|$$

$$\le\frac{1}{\Gamma_q(\alpha)}[(t_2-a)_{\omega_0}^{(\alpha-1)}-(t_1-a)_{\omega_0}^{(\alpha-1)}].$$

(b) *For* $a\le t_1\le{}_{\omega_0}\Phi_q(s)\le t_2\le b,$ *we get*

$$\left|G(t_2,{}_{\omega_0}\Phi_q(s))-G(t_1,{}_{\omega_0}\Phi_q(s))\right|$$

$$=\left|\frac{1}{\Gamma_q(\alpha)}\left[\frac{(t_2-a)_{\omega_0}^{(\alpha-1)}}{(b-a)_{\omega_0}^{(\alpha-2)}}(b-{}_{\omega_0}\Phi_q(s))_{\omega_0}^{(\alpha-2)}-(t_2-{}_{\omega_0}\Phi_q(s))_{\omega_0}^{(\alpha-1)}\right]\right.$$

$$=\frac{1}{\Gamma_q(\alpha)}\left|\frac{(b-{}_{\omega_0}\Phi_q(s))_{\omega_0}^{(\alpha-2)}}{(b-a)_{\omega_0}^{(\alpha-2)}}\left[(t_2-a)_{\omega_0}^{(\alpha-1)}-(t_1-a)_{\omega_0}^{(\alpha-1)}\right]-(t_2-{}_{\omega_0}\Phi_q(s))_{\omega_0}^{(\alpha-1)}\right|$$

$$-\frac{1}{\Gamma_q(\alpha)}\frac{(t_1-a)_{\omega_0}^{(\alpha-1)}}{(b-a)_{\omega_0}^{(\alpha-2)}}(b-{}_{\omega_0}\Phi_q(s))_{\omega_0}^{(\alpha-2)}\Bigg|$$

$$\le\frac{1}{\Gamma_q(\alpha)}[(t_2-a)_{\omega_0}^{(\alpha-1)}-(t_1-a)_{\omega_0}^{(\alpha-1)}+(t_2-t_1)_{\omega_0}^{(\alpha-1)}].$$

(c) *For* $a\le{}_{\omega_0}\Phi_q(s)\le t_1\le t_2\le b,$ *we get*

$$\left|G(t_2,{}_{\omega_0}\Phi_q(s))-G(t_1,{}_{\omega_0}\Phi_q(s))\right|$$

$$=\left|\frac{1}{\Gamma_q(\alpha)}\left[\frac{(t_2-a)_{\omega_0}^{(\alpha-1)}}{(b-a)_{\omega_0}^{(\alpha-2)}}(b-{}_{\omega_0}\Phi_q(s))_{\omega_0}^{(\alpha-2)}-(t_2-{}_{\omega_0}\Phi_q(s))_{\omega_0}^{(\alpha-1)}\right.\right.$$

$$\left.\left.-\frac{(t_1-a)_{\omega_0}^{(\alpha-1)}}{(b-a)_{\omega_0}^{(\alpha-2)}}(b-{}_{\omega_0}\Phi_q(s))_{\omega_0}^{(\alpha-2)}-(t_1-{}_{\omega_0}\Phi_q(s))_{\omega_0}^{(\alpha-1)}\right]\right|$$

$$\le\frac{1}{\Gamma_q(\alpha)}\frac{(b-{}_{\omega_0}\Phi_q(s))_{\omega_0}^{(\alpha-2)}}{(b-a)_{\omega_0}^{(\alpha-2)}}\left[\left|(t_2-a)_{\omega_0}^{(\alpha-1)}-(t_1-a)_{\omega_0}^{(\alpha-1)}\right|\right.$$

$$\left.+\left|(t_1-{}_{\omega_0}\Phi_q(s))_{\omega_0}^{(\alpha-1)}-(t_2-{}_{\omega_0}\Phi_q(s))_{\omega_0}^{(\alpha-1)}\right|\right]$$

$$\le\frac{1}{\Gamma_q(\alpha)}\left[(t_2-a)_{\omega_0}^{(\alpha-1)}-(t_1-a)_{\omega_0}^{(\alpha-1)}+(t_2-{}_{\omega_0}\Phi_q(s))_{\omega_0}^{(\alpha-1)}-(t_1-{}_{\omega_0}\Phi_q(s))_{\omega_0}^{(\alpha-1)}\right].$$

(3) $G(t,{}_{\omega_0}\Phi_q(s))\ge 0,$ *for* $(t,{}_{\omega_0}\Phi_q(s))\in(a,b)\times(a,b).$

Let $X\in C[a,b],$ the Banach space of all continuous functions on [a,b], with norm $\|u\|=\max\{|u(t):t\in[a,b]|\}.$ In our considerations, we need the standard cone $K\subset X$ by $K=\{u\in[a,b]:u(t)\ge a,a\le t\le b\}.$ It is clear that the cone K is normal.

Theorem 3.4 *Assume that*

(F_1) *there exist a real number* $d>0$ *and* $g\in L^1[a,b],$ *such that*

(i_1) $f:[a,b]\times[a,d]\to[a,+\infty)$ *is continuous,* $f(t,u)\le g(t)$ *for* $(t,u)\in[a,b]\times[a,d]$

and $f(t,u)\le f(t,v)$ *for* $a\le t\le b,a\le u\le v\le d;$

(i_2) *the following inequality holds:*

$$\frac{1}{\Gamma_q(\alpha)}\int_a^b (b-{}_{\omega_0}\Phi_q(s))_{\omega_0}^{(\alpha-2)} f\left(s, d\,\frac{(s-a)_{\omega_0}^{(\alpha-1)}}{(b-a)_{\omega_0}^{(\alpha-2)}}\right) d_{q,\omega}s \le d.$$

(F_2) *there exist* $c\in(a,d)$ *such that*

$$\int_a^b G(b, {}_{\omega_0}\Phi_q(s)) f\left(s, c\,\frac{(s-a)_{\omega_0}^{(\alpha-1)}}{(b-a)_{\omega_0}^{(\alpha-2)}}\right) d_{q,\omega}s \ge c.$$

Then the problem (1.1) *has two positive solutions* $u^*, v^* \in D$, *where*

$$D=\{u\in C[a,b] \,|\, c\,\frac{(t-a)_{\omega_0}^{(\alpha-1)}}{(b-a)_{\omega_0}^{(\alpha-2)}} \le u(t) \le d\,\frac{(t-a)_{\omega_0}^{(\alpha-1)}}{(b-a)_{\omega_0}^{(\alpha-2)}}, t\in[a,b]\}.$$

In addition, let $u_0(t)=c\,\frac{(t-a)_{\omega_0}^{(\alpha-1)}}{(b-a)_{\omega_0}^{(\alpha-2)}}$, $v_0(t)=d\,\frac{(t-a)_{\omega_0}^{(\alpha-1)}}{(b-a)_{\omega_0}^{(\alpha-2)}}$ *and construct the following sequences:*

$$u_{n+1}=\int_a^b G(t, {}_{\omega_0}\Phi_q(s)) f(s,u_n(s)) d_{q,\omega}s, \quad v_{n+1}=\int_a^b G(t, {}_{\omega_0}\Phi_q(s)) f(s,v_n(s)) d_{q,\omega}s,$$

$n=0,1,2,\ldots,$ *one has* $\lim\limits_{n\to\infty} u_n = u^*, \lim\limits_{n\to\infty} v_n = v^*.$

Proof From the non-negativeness and continuity of G and f, we can define an operator $T: C[a,b]\to C[a,b]$ by

$$Tu(t)=\int_a^b G(t, {}_{\omega_0}\Phi_q(s)) f(s,u(s)) d_{q,\omega}s, \qquad a\le t\le b.$$

From Lemma 3.1, we can see that u is the solution of problem (1.1) if and only if u is the fixed point of T. We will show that T has fixed points in the order interval $[u_0, v_0]$.

We need to show that $T:[u_0,v_0]\to C[a,b]$ is a completely continuous operator. For $u\in[u_0,v_0]$, we have $a\le c\,\frac{(t-a)_{\omega_0}^{(\alpha-1)}}{(b-a)_{\omega_0}^{(\alpha-2)}} \le u(t) \le d\,\frac{(t-a)_{\omega_0}^{(\alpha-1)}}{(b-a)_{\omega_0}^{(\alpha-2)}} \le d$, $a\le t\le b$. Since $G(t, {}_{\omega_0}\Phi_q(s))$ is continuous. So we only prove T is compact. Let $M=\int_a^b g(s) d_{q,\omega}s$, then $a\le M\le +\infty$. From the hypothesis $(F_1)-(i_2)$ and lemma 3.2, we get

$$\|Tu(t)\| = \max_{a\le t\le b}\left|\int_a^b G(t, {}_{\omega_0}\Phi_q(s)) f(s,u(s)) d_{q,\omega}s\right|$$

$$\le \max_{a\le {}_{\omega_0}\Phi_q(s)\le b} G(t, {}_{\omega_0}\Phi_q(s)) \int_a^b f(s,u(s)) d_{q,\omega}s$$

$$\le \frac{1}{\Gamma_q(\alpha)}(t-a)_{\omega_0}^{(\alpha-1)}\int_a^b g(s) d_{q,\omega}s = \frac{M}{\Gamma_q(\alpha)}(t-a)_{\omega_0}^{(\alpha-1)}.$$

This shows that the set $T([u_0,v_0])$ is uniform bounded in $C[a,b]$. After that, for given $t_1,t_2\in[a,b]$ with $t_1<t_2$, and $u\in[u_0,v_0]$, we obtain

$$|Tu_1(t)-Tu_2(t)|\le\int_a^b\left|G(t_1,{}_{\omega_0}\Phi_q(s))-G(t_2,{}_{\omega_0}\Phi_q(s))\right|f(s,u(s))d_{q,\omega}s$$

$$\le\max_{a\le{}_{\omega_0}\Phi_q(s)\le b}\left|G(t_1,{}_{\omega_0}\Phi_q(s))-G(t_2,{}_{\omega_0}\Phi_q(s))\right|\int_a^b g(s)d_{q,\omega}s$$

$$=M\max_{a\le{}_{\omega_0}\Phi_q(s)\le b}\left|G(t_1,{}_{\omega_0}\Phi_q(s))-G(t_2,{}_{\omega_0}\Phi_q(s))\right|.$$

In view of Remark 3.3(2), one has $Tu_1(t)\to Tu_2(t)$, as $t_1\to t_2$. So we claim that the set $T([u_0,v_0])$is equi-continuous in $C[a,b]$.By means of the Arzela-Ascoli theorem, $T:[u_0,v_0]\to C[a,b]$ is a completely operator. By the hypothesis $(F_1)-(i_1)$, T is an increasing operator.

From $(F_1),(F_2)$ and Lemma 3.2, for any $t\in[a,b]$, one can see that

$$Tu_0(t)=\int_a^b G(t,{}_{\omega_0}\Phi_q(s))f(s,u_0(s))d_{q,\omega}s$$

$$=\int_a^b G(t,{}_{\omega_0}\Phi_q(s))f\left(s,c\frac{(s-a)_{\omega_0}^{(\alpha-1)}}{(b-a)_{\omega_0}^{(\alpha-2)}}\right)d_{q,\omega}s$$

$$\ge\int_a^b\frac{(t-a)_{\omega_0}^{(\alpha-1)}}{(b-a)_{\omega_0}^{(\alpha-2)}}G(b,{}_{\omega_0}\Phi_q(s))f\left(s,c\frac{(s-a)_{\omega_0}^{(\alpha-1)}}{(b-a)_{\omega_0}^{(\alpha-2)}}\right)d_{q,\omega}s$$

$$\ge\frac{(t-a)_{\omega_0}^{(\alpha-1)}}{(b-a)_{\omega_0}^{(\alpha-2)}}c=u_0(t)$$

and

$$Tv_0(t)=\int_a^b G(t,{}_{\omega_0}\Phi_q(s))f(s,v_0(s))d_{q,\omega}s$$

$$=\int_a^b G(t,{}_{\omega_0}\Phi_q(s))f\left(s,d\frac{(s-a)_{\omega_0}^{(\alpha-1)}}{(b-a)_{\omega_0}^{(\alpha-2)}}\right)d_{q,\omega}s$$

$$\le\frac{1}{\Gamma_q(\alpha)}\frac{(t-a)_{\omega_0}^{(\alpha-1)}}{(b-a)_{\omega_0}^{(\alpha-2)}}\int_a^b(b-{}_{\omega_0}\Phi_q(s))_{\omega_0}^{(\alpha-2)}f\left(s,d\frac{(s-a)_{\omega_0}^{(\alpha-1)}}{(b-a)_{\omega_0}^{(\alpha-2)}}\right)d_{q,\omega}s$$

$$\le\frac{(t-a)_{\omega_0}^{(\alpha-1)}}{(b-a)_{\omega_0}^{(\alpha-2)}}d=v_0(t).$$

Hence, we get $Tu_0\ge u_0, Tv_0\le v_0$. We construct the following sequences:

$$u_{n+1}=\int_a^b G(t,{}_{\omega_0}\Phi_q(s))f(s,u_n(s))d_{q,\omega}s,\qquad v_{n+1}=\int_a^b G(t,{}_{\omega_0}\Phi_q(s))f(s,v_n(s))d_{q,\omega}s,$$

$n=0,1,2,\ldots.$ From the monotonicity of *T*, we have $u_{n+1}\geq u_n, v_{n+1}\leq v_n,$ $n=0,1,2,\ldots.$ By using Lemma 2.7, we know that the operator *T* has two positive solutions $u^*,v^*\in C[a,b]$ with $u_0\leq u^*\leq v^*\leq v_0,$ that is, $c\frac{(t-a)_{\omega_0}^{(\alpha-1)}}{(b-a)_{\omega_0}^{(\alpha-2)}}\leq u^*(t)\leq v^*(t)\leq d\frac{(t-a)_{\omega_0}^{(\alpha-1)}}{(b-a)_{\omega_0}^{(\alpha-2)}},$ $a\leq t\leq b.$ In addition, $\lim_{n\to\infty}u_n=u^*,\lim_{n\to\infty}v_n=v^*.$ □

Theorem 3.5 *Assume that*

(F_3) *there exist a real number* $d>0$ *and* $g\in L^1[a,b],$ *such that*

(i_3) $f:[a,b]\times[a,d]\to\mathbb{R}$ *is continuous,* $|f(t,u)|\leq g(t)$ *for* $(t,u)\in[a,b]\times[a,d]$ *and* $f(t,u)\leq f(t,v)$ *for* $t\in[a,b],$ $a\leq u\leq v\leq d;$

(i_4) *the following inequality holds:*

$$\frac{1}{\Gamma_q(\alpha)}\int_a^b(b-{}_{\omega_0}\Phi_q(s))_{\omega_0}^{(\alpha-2)}\max\{f\left(s,d\frac{(s-a)_{\omega_0}^{(\alpha-1)}}{(b-a)_{\omega_0}^{(\alpha-2)}}\right),0\}d_{q,\omega}s$$
$$+\int_a^b G(b,{}_{\omega_0}\Phi_q(s))\min\{f\left(s,d\frac{(s-a)_{\omega_0}^{(\alpha-1)}}{(b-a)_{\omega_0}^{(\alpha-2)}}\right),0\}d_{q,\omega}s\leq d.$$

(F_4) *there exist* $c\in[a,d]$ *such that*

$$\int_a^b G(b,{}_{\omega_0}\Phi_q(s))\max\{f\left(s,c\frac{(s-a)_{\omega_0}^{(\alpha-1)}}{(b-a)_{\omega_0}^{(\alpha-2)}}\right),0\}d_{q,\omega}s$$
$$+\frac{1}{\Gamma_q(\alpha)}\int_a^b(b-{}_{\omega_0}\Phi_q(s))_{\omega_0}^{(\alpha-2)}\min\{f\left(s,c\frac{(s-a)_{\omega_0}^{(\alpha-1)}}{(b-a)_{\omega_0}^{(\alpha-2)}}\right),0\}d_{q,\omega}s\geq c.$$

Then the problem (1.1) *has two positive solutions* $u^*,v^*\in D,$ *where*

$$D=\{u\in C[a,b]\,\Big|\,c\frac{(t-a)_{\omega_0}^{(\alpha-1)}}{(b-a)_{\omega_0}^{(\alpha-2)}}\leq u(t)\leq d\frac{(t-a)_{\omega_0}^{(\alpha-1)}}{(b-a)_{\omega_0}^{(\alpha-2)}},t\in[a,b]\}.$$

In addition, let $u_0(t)=c\frac{(t-a)_{\omega_0}^{(\alpha-1)}}{(b-a)_{\omega_0}^{(\alpha-2)}},$ $v_0(t)=d\frac{(t-a)_{\omega_0}^{(\alpha-1)}}{(b-a)_{\omega_0}^{(\alpha-2)}}$ *and construct the following sequences:*

$$u_{n+1}=\int_a^b G(t,{}_{\omega_0}\Phi_q(s))f(s,u_n(s))d_{q,\omega}s,\quad v_{n+1}=\int_a^b G(t,{}_{\omega_0}\Phi_q(s))f(s,v_n(s))d_{q,\omega}s,$$

$n=0,1,2,\ldots,$ *one has* $\lim_{n\to\infty}u_n=u^*,\lim_{n\to\infty}v_n=v^*.$

Proof Consider the same operator $T:C[a,b]\to C[a,b]$ as defined in the proof of Theorem 3.4:

$$Tu(t)=\int_a^b G(t,{}_{\omega_0}\Phi_q(s))f(s,u(s))d_{q,\omega}s,\qquad t\in[a,b].$$

We also show that T has fixed points in the order interval $[u_0, v_0]$.

Similar to the proof of Theorem 3.4, $T : C[a,b] \to C[a,b]$ is a completely continuous operator. From the hypothesis $(F_3)-(i_3)$, T is an increasing operator. Further, by using the conditions $(F_3),(F_4)$, Remark 3.3 and Lemma 3.2, for any $t \in [a,b]$, one obtains

$$\begin{aligned}
Tu_0(t) &= \int_a^b G(t, {}_{\omega_0}\Phi_q(s)) f(s, u_0(s)) d_{q,\omega}s \\
&= \int_a^b G(t, {}_{\omega_0}\Phi_q(s)) \max\{ f\left(s, c\frac{(s-a)_{\omega_0}^{(\alpha-1)}}{(b-a)_{\omega_0}^{(\alpha-2)}} \right), 0\} d_{q,\omega}s \\
&\quad + \int_a^b G(t, {}_{\omega_0}\Phi_q(s)) \min\{ f\left(s, c\frac{(s-a)_{\omega_0}^{(\alpha-1)}}{(b-a)_{\omega_0}^{(\alpha-2)}} \right), 0\} d_{q,\omega}s \\
&\ge \frac{(t-a)_{\omega_0}^{(\alpha-1)}}{(b-a)_{\omega_0}^{(\alpha-2)}} \left[\int_a^b G(b, {}_{\omega_0}\Phi_q(s)) \max\{ f\left(s, c\frac{(s-a)_{\omega_0}^{(\alpha-1)}}{(b-a)_{\omega_0}^{(\alpha-2)}} \right), 0\} d_{q,\omega}s \right. \\
&\quad \left. + \frac{1}{\Gamma_q(\alpha)} \int_a^b (b - {}_{\omega_0}\Phi_q(s))_{\omega_0}^{(\alpha-2)} \min\{ f\left(s, c\frac{(s-a)_{\omega_0}^{(\alpha-1)}}{(b-a)_{\omega_0}^{(\alpha-2)}} \right), 0\} d_{q,\omega}s \right] \\
&\ge \frac{(t-a)_{\omega_0}^{(\alpha-1)}}{(b-a)_{\omega_0}^{(\alpha-2)}} c = u_0(t)
\end{aligned}$$

and

$$\begin{aligned}
Tv_0(t) &= \int_a^b G(t, {}_{\omega_0}\Phi_q(s)) f(s, v_0(s)) d_{q,\omega}s \\
&= \int_a^b G(t, {}_{\omega_0}\Phi_q(s)) \max\{ f\left(s, d\frac{(s-a)_{\omega_0}^{(\alpha-1)}}{(b-a)_{\omega_0}^{(\alpha-2)}} \right), 0\} d_{q,\omega}s \\
&\quad + \int_a^b G(t, {}_{\omega_0}\Phi_q(s)) \min\{ f\left(s, d\frac{(s-a)_{\omega_0}^{(\alpha-1)}}{(b-a)_{\omega_0}^{(\alpha-2)}} \right), 0\} d_{q,\omega}s \\
&\le \frac{(t-a)_{\omega_0}^{(\alpha-1)}}{(b-a)_{\omega_0}^{(\alpha-2)}} d = v_0(t).
\end{aligned}$$

Hence, we get $Tu_0 \ge u_0, Tv_0 \le v_0$. We construct the following sequences:

$$u_{n+1} = \int_a^b G(t, {}_{\omega_0}\Phi_q(s)) f(s, u_n(s)) d_{q,\omega}s, \qquad v_{n+1} = \int_a^b G(t, {}_{\omega_0}\Phi_q(s)) f(s, v_n(s)) d_{q,\omega}s,$$

$n = 0,1,2,\ldots.$ According to the monotonicity of *T*, we get $u_{n+1} \ge u_n, v_{n+1} \le v_n, \quad n = 0,1,2,\ldots.$

By using Lemma 2.7, we know that the operator T has two positive solutions $u^*, v^* \in C[a,b]$ with $u_0 \le u^* \le v^* \le v_0$, that is, $a < c\frac{(t-a)_{\omega_0}^{(\alpha-1)}}{(b-a)_{\omega_0}^{(\alpha-2)}} \le u^*(t) \le v^*(t) \le d\frac{(t-a)_{\omega_0}^{(\alpha-1)}}{(b-a)_{\omega_0}^{(\alpha-2)}} \le d, \quad a < t \le b.$

In addition, $\lim_{n\to\infty} u_n = u^*, \lim_{n\to\infty} v_n = v^*.$

By using the same proof as Theorem 3.5, we can easily obtain the following conclusion.

Theorem 3.6 *Assume that*

(F_5) *there exist a real number* $c<0$ *and* $g \in L^1[a,b]$, *such that*

(i_5) $f:[a,b]\times[c,a]\to\mathbb{R}$ *is continuous,* $|f(t,u)| \le g(t)$ *for* $(t,u)\in[a,b]\times[c,a]$ *and* $f(t,u) \le f(t,v)$ *for* $t\in[a,b]$, $c \le u \le v \le a.$

In addition, there exist $d \in (c,a)$ *such that* $(F_3)-(i_4)$ *and* (F_4) *in* Theorem 3.5 *are also satisfied. Then the problem* (1.1) *has two negative solutions* $u^*, v^* \in D$, *where*

$$D = \{u \in C[a,b] \Big| c\frac{(t-a)_{\omega_0}^{(\alpha-1)}}{(b-a)_{\omega_0}^{(\alpha-2)}} \le u(t) \le d\frac{(t-a)_{\omega_0}^{(\alpha-1)}}{(b-a)_{\omega_0}^{(\alpha-2)}}, t\in[a,b]\}$$

Let $u_0(t) = c\frac{(t-a)_{\omega_0}^{(\alpha-1)}}{(b-a)_{\omega_0}^{(\alpha-2)}}$, $v_0(t) = d\frac{(t-a)_{\omega_0}^{(\alpha-1)}}{(b-a)_{\omega_0}^{(\alpha-2)}}$ *and we construct the following sequences:*

$$u_{n+1} = \int_a^b G(t, {}_{\omega_0}\Phi_q(s)) f(s, u_n(s)) d_{q,\omega}s, \quad v_{n+1} = \int_a^b G(t, {}_{\omega_0}\Phi_q(s)) f(s, v_n(s)) d_{q,\omega}s,$$

$n = 0,1,2,\ldots,$ *we can obtain* $\lim_{n\to\infty} u_n = u^*, \lim_{n\to\infty} v_n = v^*.$

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